Computers and Mathematics with Applications 61 (2011) 1682–1686

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

An algorithm for solving the general variational inclusion involving A-monotone operators

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ARTICLE INFO

Article history:
Received 7 October 2010
Accepted 27 January 2011

Keywords:
Resolvent–Projection algorithm
A-monotone operator
Variational inclusion
Hilbert space

ABSTRACT

In this paper, the Resolvent–Projection algorithm for solving the variational inclusion $0 \in M(x)$ involving an A-monotone set-valued operator M is constructed in Hilbert spaces. The convergence of the iterative sequence generated by the algorithm is proved also.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, and $2^{\mathcal{H}}$ denote the family of all the nonempty subsets of $\mathcal{H}$. Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued mapping, and S denote the root set of M, i.e., $S = \{x \in \mathcal{H} : 0 \in M(x)\}$. We consider a class of nonlinear variational inclusion problems: find $x \in \mathcal{H}$ such that

$$0 \in M(x). \tag{1.1}$$

Throughout this paper, we assume that $S \neq \emptyset$. As a matter of fact, a general class of problems of minimization or maximization of functions, variational inequality problems, and minimax problems can be unified into the form (1.1) (see Refs. [1–6]). A fundamental algorithm for finding a solution of Problem (1.1) is the proximal point algorithm given by Rockafellar [2] in 1976. For when M is maximal monotone, in [2], Rockafellar investigated the general convergence and rate of convergence for the algorithm in the context of solving (1.1) by showing that the sequence $\{x^k\}$ generated for an initial point x^0 by

$$x^{k+1} \approx J_k(x^k) \tag{1.2}$$

converges weakly to a solution to (1.1), provided the approximation is made sufficiently accurate as the iteration proceeds, where $J_k = (I + \lambda_k M)^{-1}$ for a sequence $\{\lambda_k\}$ of positive real numbers that is bounded away from zero.

Recently, applying the idea of Rockafellar [2], Verma [7] presented a algorithm for approximating a solution to Problem (1.1) involving a set-valued A-maximal monotone mapping M in a Hilbert space setting, and showed the following convergence theorem.

Theorem 1.1. *Let $\mathcal{H}$ be a real Hilbert space, let $A : \mathcal{H} \to \mathcal{H}$ be r-strongly monotone and s-Lipschitz continuous, and let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be A-maximal monotone. For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by the proximal point algorithm*

$$x^{k+1} \approx J_{A,k}(A(x^k)), \tag{1.3}$$

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doi:10.1016/j.camwa.2011.01.039

such that

$$\|x^{k+1} - J_{A,k}(A(x^k))\| \le \varepsilon_k,$$

where $J_{A,k} = (A + \lambda_k M)^{-1}$*, and* $\{\varepsilon_k\}$*,* $\{\lambda_k\} \subset [0, 1)$ *are scalar sequences with* $e_1 = \sum_{k=0}^{\infty} \varepsilon_k < \infty$*, and* λ_k *is bounded away from zero. Then the sequence* $\{x^k\}$ *converges weakly to a solution of Problem (1.1).*

Note that there is no efficient and implementable method for searching for the proximal point in the algorithms mentioned above, and it is of interest and importance to develop an efficient and implementable algorithm for solving Problem (1.1) and its generalizations (see Refs. [8–14]). In this paper, we first construct a new iterative algorithm, which consists of a resolvent operator technique step followed by a suitable orthogonal projection onto a moving hyperplane, for approximating the solution of Problem (1.1), involving a set-valued A-maximal monotone operator M in Hilbert space as follows:

Algorithm 1.1 (*Resolvent–Projection Algorithm*)**.**

Step 0. (Initiation) Select initial $z^0 \in \mathcal{H}$ and set $k = 0$.
Step 1. (Resolvent step) Find $x^k \in \mathcal{H}$ such that

$$x^k = J_{A,k}(A(z^k)), \tag{1.4}$$

where a positive sequence $\{\lambda_k\}$ satisfies $\alpha := \inf_{k\ge 0} \lambda_k > 0$.
Step 2. (Projection step) Set $K = \{z \in \mathcal{H} : \langle A(z^k) - A(x^k), z - A(x^k)\rangle \le 0\}$. If $A(x^k) = A(z^k)$, then stop; otherwise, take z^{k+1} such that

$$A(z^{k+1}) = P_K(A(z^k)), \tag{1.5}$$

where $P_K(A(z^k))$ denotes the projection of $A(z^k)$ onto K.
Step 3. Let $k = k + 1$ and return to Step 1.

We also prove that the iterative sequence $\{x^k\}$ is weakly convergent to a solution of Problem (1.1).

2. Preliminaries

Suppose that $X \subset \mathcal{H}$ is a nonempty closed convex subset and

$$\mathrm{dist}(z, X) := \inf_{x\in X} \|z - x\|$$

is the distance from z to X. Let $P_X[z]$ denote the projection of z onto X, that is, $P_X[z]$ satisfies the condition

$$\|z - P_X[z]\| = \mathrm{dist}(z, X).$$

The following well-known properties of the projection operator will be used in this paper.

Proposition 2.1 ([5])**.** *Let X be a nonempty closed convex subset in $\mathcal{H}$. Then,*

$$u = P_X[x] \iff \langle u - x, y - u\rangle \ge 0, \quad \text{for all } x \in \mathcal{H} \text{ and } y \in X.$$

Definition 2.1. Let $A : \mathcal{H} \to \mathcal{H}$ be a single-valued operator. A is said to be:

(i) monotone if $\langle Ax - Ay, x - y\rangle \ge 0$, $\forall x, y \in \mathcal{H}$;
(ii) strictly monotone if A is monotone and $\langle Ax - Ay, x - y\rangle = 0$ if and only if $x = y$;
(iii) ζ-strongly monotone if there exists constant $\zeta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \zeta\|x - y\|^2, \quad \forall x, y \in \mathcal{H}.$$

Definition 2.2. Let $A : \mathcal{H} \to \mathcal{H}$ be a single-valued operator. A multi-valued operator M is said to be:

(i) monotone if

$$\langle u - v, x - y\rangle \ge 0, \quad \forall x, y \in \mathcal{H}, u \in Mx, v \in My;$$

(ii) monotone with respect to A if

$$\langle u - v, Ax - Ay\rangle \ge 0, \quad \forall x, y \in \mathcal{H}, u \in Mx, v \in My;$$

(iii) maximal monotone if M is monotone and $(I + \lambda M)(\mathcal{H}) = \mathcal{H}$ for all $\lambda > 0$, where I denotes the identity mapping on $\mathcal{H}$;

(iv) relaxed monotone if there exists a positive constant ξ such that

$$\langle u - v, x - y \rangle \geq -\xi \|x - y\|^2, \quad \forall x, y \in \mathcal{H}, u \in Mx, v \in My;$$

(v) A-monotone [15] if M is relaxed monotone and $(A + \lambda M)(\mathcal{H}) = \mathcal{H}$ holds for every $\lambda > 0$.

Definition 2.3 (*[16]*). Let $A : \mathcal{H} \to \mathcal{H}$ be a strictly monotone mapping and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be an A-monotone mapping. The resolvent operator $R^A_{M,\lambda} : \mathcal{H} \to \mathcal{H}$ is defined by

$$R^A_{M,\lambda}(x) = (A + \lambda M)^{-1}(x), \quad \forall x \in \mathcal{H}. \tag{2.1}$$

Lemma 2.1 (*[17]*). *Let $A : \mathcal{H} \to \mathcal{H}$ be a strongly monotone single-valued mapping and $M : \mathcal{H} \to 2^{\mathcal{H}}$ an A-monotone mapping. Then M is maximal monotone.*

3. Convergence of the Resolvent–Projection Algorithm

In this section, we prove that the iterative sequence $\{x^k\}$ generated by the Resolvent–Projection Algorithm 1.1 is weakly convergent to a solution of Problem (1.1).

Proposition 3.1. *Resolvent–Projection Algorithm 1.1 is well-defined.*

Proof. From Lemma 2.1, it follows that M is maximal monotone. For any $\lambda_k > 0$, M is maximal monotone if and only if $\lambda_k M$ is maximal monotone. So, by Theorem 1 in [18], M is maximal monotone if and only if $(I + \lambda_k M)(\mathcal{H}) = \mathcal{H}$. Therefore, it follows from Lemma 3 of [19] that there is a unique $x^k \in \mathcal{H}$ such that (1.4) holds. Obviously, the set K is nonempty closed convex subset of $\mathcal{H}$. When $x^k = z^k$, by (1.4), z^k is a solution of Problem (1.1). If the iterative sequence $\{x^k\}$ is finite, then the last term is a solution of Problem (1.1). Therefore Algorithm 1.1 is well-defined. □

Theorem 3.1. *Suppose that $A : \mathcal{H} \to \mathcal{H}$ is a ζ-strongly monotone, continuous single-valued mapping, and a set-valued A-monotone mapping $M : \mathcal{H} \to 2^{\mathcal{H}}$ is monotone with respect to A. Then the infinite sequence $\{x^k\}$ generated by Algorithm 1.1 is convergent weakly to a solution of Problem (1.1).*

Proof. Suppose that $x^* \in \mathcal{H}$ is a solution of problem (1.1). We divide the proof of Theorem 3.1 into four steps.

Step 1. We first show that $A(x^*) \in K$. Since $x^* \in S$, we have $0 \in M(x^*)$. From (1.4), it follows that

$$\frac{1}{\lambda_k}(A(z^k) - A(x^k)) \in M(x^k).$$

By the monotonicity with respect to A of M, we deduce that

$$\left\langle 0 - \frac{1}{\lambda_k}(A(z^k) - A(x^k)), A(x^*) - A(x^k) \right\rangle \geq 0,$$

which leads to

$$A(x^*) \in K = \{z \in H : \langle A(z^k) - A(x^k), z - A(x^k) \rangle \leq 0\}.$$

Step 2. We show that the infinite sequence $\{x^k\}$ generated by Algorithm 1.1 is bounded. Since $A(z^{k+1}) = P_K(A(z^k))$, by Proposition 2.1 we deduce that

$$\langle A(z^{k+1}) - A(z^k), A(x^*) - A(z^{k+1}) \rangle \geq 0.$$

From

$$\begin{aligned} \|A(x^*) - A(z^{k+1})\|^2 &= \|A(x^*) - A(z^k) - (A(z^{k+1}) - A(z^k))\|^2 \\ &= \|A(x^*) - A(z^k)\|^2 - 2\langle A(x^*) - A(z^k), A(z^{k+1}) - A(z^k) \rangle + \|A(z^{k+1}) - A(z^k)\|^2 \\ &= \|A(x^*) - A(z^k)\|^2 - 2\langle A(z^{k+1}) - A(z^k), A(z^{k+1}) - A(z^k) \rangle \\ &\quad - 2\langle A(x^*) - A(z^{k+1}), A(z^{k+1}) - A(z^k) \rangle + \|A(z^{k+1}) - A(z^k)\|^2 \\ &\leq \|A(x^*) - A(z^k)\|^2 - 2\langle A(x^*) - A(z^{k+1}), A(z^{k+1}) - A(z^k) \rangle - \|A(z^{k+1}) - A(z^k)\|^2, \end{aligned}$$

we have

$$\|A(x^*) - A(z^{k+1})\|^2 \leq \|A(x^*) - A(z^k)\|^2 - \|A(z^{k+1}) - A(z^k)\|^2. \tag{3.1}$$

Thus,

$$\|A(x^*) - A(z^{k+1})\| \leq \|A(x^*) - A(z^k)\|, \quad \forall k \geq 0,$$

which yields that the sequence $\{\|A(x^*) - A(z^k)\|\}$ is convergent. From the strong monotonicity of A, we have that

$$\frac{1}{\zeta}\|A(x^*) - A(z^k)\| \geq \|x^* - z^k\|.$$

Hence, the infinite sequence $\{z^k\}$ generated by Algorithm 1.1 is bounded. It follows from (3.1) that

$$0 \leq \|A(z^{k+1}) - A(z^k)\|^2 \leq \|A(x^*) - A(z^k)\|^2 - \|A(x^*) - A(z^{k+1})\|^2 \tag{3.2}$$

and so (3.2) implies that

$$\lim_{k\to\infty}\|A(z^{k+1}) - A(z^k)\|^2 \leq \lim_{k\to\infty}[\|A(x^*) - A(z^k)\|^2 - \|A(x^*) - A(z^{k+1})\|^2] = 0.$$

Thus, we know that $\lim_{k\to\infty}\|A(z^{k+1}) - A(z^k)\| = 0$ holds.

From $A(z^{k+1}) = P_K(A(z^k)) \in K$ and $A(z^k) \notin K$, we have that

$$\langle A(z^k) - A(x^k), A(z^{k+1}) - A(x^k)\rangle \leq 0,$$

and

$$\|A(x^k) - A(z^k)\|^2 = \langle A(x^k) - A(z^k), A(x^k) - A(z^k)\rangle \leq \langle A(z^{k+1}) - A(z^k), A(x^k) - A(z^k)\rangle.$$

Therefore,

$$\lim_{k\to\infty}\|A(z^k) - A(x^k)\| = 0. \tag{3.3}$$

Since A is ζ-strongly monotone, we have that

$$\|A(x^k) - A(z^k)\|\,\|x^k - z^k\| \geq \langle A(x^k) - A(z^k), x^k - z^k\rangle \geq \zeta\|x^k - z^k\|^2.$$

Hence, $\lim_{k\to\infty}\|z^k - x^k\| = 0$, and $\lim_{k\to\infty}(z^k - x^k) = 0$. This implies that $\{x^k\}$ is bounded also. Moreover, $\{x^k\}$ and $\{z^k\}$ have the same weak accumulation points.

Step 3. We now show that every weak accumulation point of the sequence $\{x^k\}$ generated by Algorithm 1.1 is a solution of problem (1.1).

Let $\hat{x}$ be a weak accumulation point of $\{x^k\}$. We can extract a subsequence that weakly converges to $\hat{x}$. Without loss of generality, let us suppose that $\lim_{k\to\infty} x^k = \hat{x}$ (weakly). Then we have $\lim_{k\to\infty} z^k = \hat{x}$ (weakly). For any fixed $v \in \mathcal{H}$, take an arbitrary $u \in M(v)$. Then, it follows from the monotonicity of M that

$$\left\langle x^k - v, \frac{1}{\lambda_k}(A(z^k) - A(x^k)) - u\right\rangle \geq 0,$$

and

$$\langle x^k - v, -u\rangle \geq -\left\langle x^k - v, \frac{1}{\lambda_k}(A(z^k) - A(x^k))\right\rangle. \tag{3.4}$$

By (3.3), the boundedness of $\{x^k\}$ and $\{\lambda_k\}$, we have

$$\left\langle x^k - v, \frac{1}{\lambda_k}(A(z^k) - A(x^k))\right\rangle \leq \frac{1}{\alpha}\|x^k - v\| \cdot \|A(z^k) - A(x^k)\| \to 0 \quad (k \to \infty).$$

Taking limits in (3.4),

$$\langle \hat{x} - v, 0 - u\rangle = \lim_{k\to\infty}\langle x^k - v, 0 - u\rangle \geq 0.$$

By Lemma 2.1, we know that $M : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone mapping. Since (v, u) is an arbitrary point in the graph of M, i.e., $\mathrm{Graph}(M) = \{(v, u) : u \in M(v)\}$, we conclude that $(\hat{x}, 0) \in \mathrm{Graph}(M)$ and $0 \in M(\hat{x})$. This shows that $\hat{x} \in X$ is a solution of problem (1.1).

Step 4. We show that the sequence $\{x^k\}$ generated by Algorithm 1.1 has a unique weak accumulation point, and $\{x^k\}$ is weakly convergent.

Let $\hat{z}$ and $\bar{z}$ be two weak accumulation points of $\{z^k\}$, and $\{z^{k_j}\}$ and $\{z^{k_i}\}$ be two subsequences of $\{z^k\}$ that weakly converge to $\hat{z}$ and $\bar{z}$, respectively. Then $\hat{z}, \bar{z} \in S$ and the sequences $\{\|A(z^k) - A(\hat{z})\|^2\}$ and $\{\|A(z^k) - A(\bar{z})\|^2\}$ are convergent. Let $\xi = \lim_{k\to\infty}\|A(z^k) - A(\hat{z})\|^2$, $\eta = \lim_{k\to\infty}\|A(z^k) - A(\bar{z})\|^2$ and $\gamma = \|A(\hat{z}) - A(\bar{z})\|^2$; then

$$\|A(z^{k_j}) - A(\bar{z})\|^2 = \|A(z^{k_j}) - A(\hat{z})\|^2 + \|A(\hat{z}) - A(\bar{z})\|^2 + 2\langle A(z^{k_j}) - A(\hat{z}), A(\hat{z}) - A(\bar{z})\rangle \tag{3.5}$$

and

$$\|A(z^{k_i}) - A(\hat{z})\|^2 = \|A(z^{k_i}) - A(\bar{z})\|^2 + \|A(\hat{z}) - A(\bar{z})\|^2 + 2\langle A(z^{k_i}) - A(\bar{z}), A(\bar{z}) - A(\hat{z})\rangle. \tag{3.6}$$

We take the limit in (3.5) as $j \to \infty$ and (3.6) as $i \to \infty$, observing that the inner products in the right hand sides of (3.5) and (3.6) converge to 0 because A is continuous and $\hat{z}$, $\bar{z}$ are weak limits of $\{z^{k_j}\}$, $\{z^{k_i}\}$ respectively. From the definitions of ξ, η, γ, it follows that

$$\xi = \eta + \gamma, \qquad \eta = \xi + \gamma,$$

and $\xi - \eta = \gamma = \eta - \xi$, which implies $\gamma = 0$, i.e. $A(\hat{z}) = A(\bar{z})$. Since A is ζ-strongly monotone,

$$\zeta \|\hat{z} - \bar{z}\|^2 \le \langle A(\hat{z}) - A(\bar{z}), \hat{z} - \bar{z}\rangle \le \|A(\hat{z}) - A(\bar{z})\| \, \|\hat{z} - \bar{z}\|. \tag{3.7}$$

It follows from (3.7) and $A(\hat{z}) = A(\bar{z})$ that

$$\hat{z} = \bar{z}.$$

We conclude that all weak accumulation points of $\{z^k\}$ coincide, i.e., $\{z^k\}$ is weakly convergent. Thus, $\{x^k\}$ is weakly convergent to a solution of Problem (1.1). This completes the proof. □

Theorem 3.2. *Let M be a maximal monotone mapping on a Hilbert space $\mathcal{H}$, and the iterative sequence $\{x^k\}$ be defined by the following algorithm:*

Step 0. *(Initiation) Select initial $z^0 \in \mathcal{H}$ and set $k = 0$.*
Step 1. *(Resolvent step) Find $x^k \in \mathcal{H}$ such that*

$$x^k = J_k(z^k), \tag{3.8}$$

where a positive sequence $\{\lambda_k\}$ satisfies $\alpha_1 := \inf_{k \ge 0} \lambda_k > 0$.
Step 2. *(Projection step) Set $K = \{z \in \mathcal{H} : \langle z^k - x^k, z - x^k\rangle \le 0\}$. If $x^k = z^k$, then stop; otherwise, take*

$$z^{k+1} = P_K(z^k), \tag{3.9}$$

where $P_K(z^k)$ denotes the projection of z^k onto K.
Step 3. *Let $k = k + 1$ and return to Step* 1.

Then $\{x^k\}$ is weakly convergent to a solution of Problem (1.1).

Acknowledgements

This work was supported by a grant from the "Project 211 (Phase III)" of the Southwestern University of Finance and Economics (No. QN09-106) and the Scientific Research Fund of the Southwestern University of Finance and Economics (No. 10XG052).

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